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LETTER TO EDITOR

Lattice decorations and pseudo-continuum percolation

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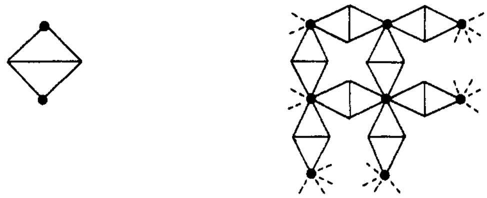
Abstract. Several families of two-dimensional lattices are discussed for which the critical percolation density (p_c) can be calculated exactly. The lattices are obtained by decorating lattices for which the value of the critical density for bond percolation is already known. It is shown that finite decorations of this type do not change the value of a critical exponent. By a sequence of decorations and lattice transformations we obtain a set of lattices of increasing coordination number q, and calculate the limit qp_c for $q \to \infty$ for this family. The value of this limit is close to the numerical estimate of the corresponding critical density for continuum percolation.

For any lattice L we consider a decorated lattice L^D obtained from L in the following way. The bond set of L is replaced by a set of identical two-rooted graphs (with a finite number of vertices) where the roots replace the two vertices incident on the original bond in L, and each pair of two-rooted graphs has no vertices in common, except possibly their roots. For instance, if L is the square lattice and the decoration is the two-rooted graph in figure 1(a), L^D is the lattice in figure 1(b).



(a) (b)

Figure 1. Decorating the square lattice with the two-rooted graph (a) leads to the lattice (b).

If we consider the bond percolation problem on this pair of lattices we can map any configuration on L^D onto a corresponding (unique) configuration on L by occupying a bond A–B in L if and only if the roots in the corresponding decoration on L^D are connected. This surjection of configurations in L^D onto configurations in L clearly preserves percolation in that if a configuration C percolates on L^D then its image under this surjection percolates on L.

If the bond density on L^D is p then the probability that the roots in the decoration are connected, $f(p)$, will be a non-decreasing function of p. For instance, for the example in figure 1

$$f(p) = 2p^2(1-p)^3 + 2p^3(1-p)^2 + 5p^4(1-p) + p^5. \tag{1}$$

If the critical bond density on L is $p_c(\mathrm{L})$ and the corresponding critical density on L^{D} is $p_c(\mathrm{L}^{\mathrm{D}})$, we have

$$p_c(\mathrm{L}) = f(p_c(\mathrm{L}^{\mathrm{D}})). \tag{2}$$

To relate the exponents on the two lattices it is convenient to write the bond density and critical bond density on L^{D} as p and p_c, and the corresponding quantities on L as f and f_c. The percolation probability on L is $F(f)$ which we assume to have the power law behaviour

$$F(f) \sim A(f-f_c)^{\beta}, \qquad f \to f_c+. \tag{3}$$

For any percolating configuration on L^{D} the root points have identical connectivity to their images in L, so that their percolation probability is preserved. The percolation probability for non-root points in L^{D} is just the probability of being connected to at least one root times the root percolation probability. Since the probability of being connected to at least one root times the root percolation probability of being connected to at least one root is a finite polynomial and hence analytic, the percolation probability on L^{D} is then of the form

$$P(p) \sim B(f-f_c)^{\beta} \sim B[f'(p_c)(p-p_c)]^{\beta} \tag{4}$$

provided that $f'(p_c) \neq 0$. That is, with this condition the critical exponent is unchanged by the decoration. To show that $f'(p_c) \neq 0$ we argue as follows: If the roots on a two-rooted graph are connected then there exists at least one self-avoiding walk joining the roots A and B. Consider any bond j on such a walk and write

$$\begin{aligned} f(p) &= \Pr\{\mathrm{AB}|j\}p_j + \Pr\{\mathrm{AB}|\bar{j}\}(1-p_j) \\ &= \Pr\{\mathrm{AB}|\bar{j}\} + (\Pr\{\mathrm{AB}|j\} - \Pr\{\mathrm{AB}|\bar{j}\})p_j \end{aligned} \tag{5}$$

where $\Pr\{\mathrm{AB}|j\}$ and $\Pr\{\mathrm{AB}|\bar{j}\}$ are the conditional probabilities that A and B are connected given that bond j is or is not present, respectively, and p_j is the probability that bond j is present. Since $\Pr\{\mathrm{AB}|j\}$ and $\Pr\{\mathrm{AB}|\bar{j}\}$ are not explicit functions of p_j,

$$\partial f(p)/\partial p_j = \Pr\{\mathrm{AB}|j\} - \Pr\{\mathrm{AB}|\bar{j}\}. \tag{6}$$

This quantity is non-negative since adding a bond cannot decrease a percolation probability. If $\partial f(p)/\partial p_j = 0$ then $\Pr\{\mathrm{AB}\}$ is independent of bond j and this can only occur if either

(i) there is at least one self-avoiding walk connecting the roots which has all bonds present with probability one, or

(ii) every self-avoiding walk containing j and connecting the roots has at least one bond absent with probability one.

Neither of these is possible provided that we restrict our attention to bond probabilities in the open interval. Since $\mathrm{d}f/\mathrm{d}p$ may therefore be expanded as a sum of non-negative terms, and at least one such term is positive, $\mathrm{d}f/\mathrm{d}p > 0$ for $0 < p < 1$. Hence these finite decorations do not alter the exponent β. Unfortunately, we have been unable to extend these arguments to infinite decorations.

Considering pseudo-continuum percolation, Domb (1972) discussed a family of lattice models in which the range of percolation extends to first, second, third, etc neighbours, so that the coordination number (g) in the site percolation process can in principle be increased indefinitely. The corresponding critical site densities (p_c) were estimated by series analysis and an estimate was made of

$$L = \lim_{q \to \infty} q p_c$$

in order to make contact with continuum percolation (Gilbert 1961). In the same spirit we use bond decorations to construct a family of lattices, L^D, with decreasing average coordination numbers, construct the corresponding covering lattices, L^C, (to convert to site percolation) and then find the corresponding matching lattices, L^m. If the critical density is known for the bond problem on the original lattice, the critical site density (p_c) can be calculated on L^m and the value of $\lim_{q\to\infty} qp_c$ where q is the coordination number of the matching lattice, can be calculated exactly.

For example, for the bond problem on the square lattice (L)$p_c = \frac{1}{2}$ (Sykes and Essam 1964). To construct L^D we replace each bond on L by n bonds in series (figure 2(a)) and then carry out a bond–site transformation to obtain L^c (figure 2(b)). L^D has the connectivity function

$$f(p) = p^n \tag{7}$$

so that the critical density for bond percolation on L^D (and site percolation on L^c) is $(\frac{1}{2})^{1/n}$. We now construct the matching lattice of L^c (figure 2(c)) which has critical density for site percolation (p_c) given by

$$p_c = 1 - (\tfrac{1}{2})^{1/n} \tag{8}$$

and coordination number

$$q = 7n - 1. \tag{9}$$

Letting $n \to \infty$ we have

$$L = \lim_{n\to\infty} qp_c = 7 \ln 2 = 4{\cdot}852\ldots \tag{10}$$

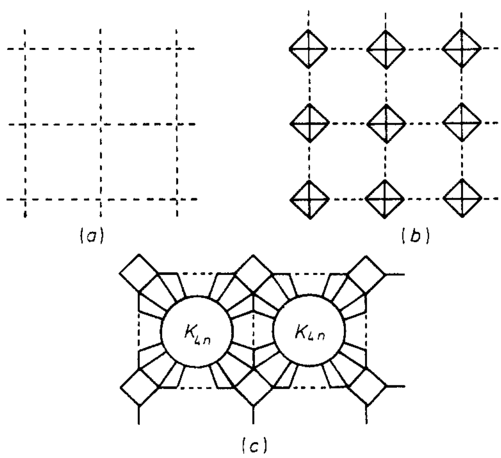


Figure 2. (a) A square lattice in which each bond is replaced by n bonds in series. (b) The covering lattice of (a). (c) The matching lattice of (b) in which K_{4n} is the complete graph on $4n$ vertices.

Similar calculations for the hexagonal and triangular lattices give $L = 4{\cdot}692\ldots$ and $5{\cdot}287\ldots$ respectively.

These values of L clearly depend on the parent lattice but are all reasonably close to Domb's estimate for the continuum of 4·0 to 4·5 (Domb 1972). The limiting lattices differ from the continuum in a number of important ways. Clearly the lattice points are countable but, more importantly, they are not dense in the plane. In spite of this, they represent examples of lattices with infinite coordination number, for which L can be calculated exactly and which give values of L close to that of the continuum.

In order to understand why the calculated values are higher than the continuum value consider an arbitrary point on the lattice shown in figure 2(c). Construct the smallest disc centred at this point which just contains all the $7n-1$ 'neighbours' of the initial point. In general the disc will contain other lattice points to which the initial point is not connected so that in this sense the lattice has lower connectivity than the corresponding continuum problem and would be expected to percolate at higher density. In addition the lattice points are distributed along a set of lines (rather than Poisson distributed in the plane) so that discs centred at these lattice points will 'cover' the plane less efficiently.

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